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Mirror Symmetry and the Web of Landau-Ginzburg String Vacua

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ABSTRACT

We present some mathematical aspects of Landau-Ginzburg string vacua in terms of toric geometry. The one-to-one correspondence between toric divisors and some of $(-1,1)$ states in Landau-Ginzburg model is presented for superpotentials of typical types. The Landau-Ginzburg interpretation of non-toric divisors is also presented. Using this interpretation, we propose a method to solve the so-called "twisted sector problem" by orbifold construction. Moreover,this construction shows that the moduli spaces of the original Landau-Ginzburg string vacua and their orbifolds are connected. By considering the mirror map of Landau-Ginzburg models, we obtain the relation between Mori vectors and the twist operators of our orbifoldization. This consideration enables us to argue the embedding of the Seiberg-Witten curve in the defining equation of the Calabi-Yau manifolds on which the type II string gets compactified. Related topics concerning the Calabi-Yau fourfolds and the extremal transition are discussed.

1 Introduction

Strings on Calabi-Yau manifolds are promising candidates for unified theory. Their physics heavily depends on the topological properties of Calabi-Yau manifolds. Recently it is believed that the toric geometry gives us the most useful technique to study Calabi-Yau manifolds [1, 2].

It is also known that Landau-Ginzburg models [3] can describe the strings on Calabi-Yau manifolds. They give us a powerful method for calculating the (anti-)chiral rings [4] which can be identified with the cohomology rings on the corresponding Calabi-Yau manifolds, i.e.(p, q) states of Landau-Ginzburg model can be identified with $(3-p, q)$ forms on a Calabi-Yau manifold.

In this paper, we study the relation between Landau-Ginzburg models and toric geometry. First, we find the one-to-one identification between a toric divisor and a $(-1, 1)$ state which is a ground state in j^{-l} twisted sector, where j is the $U(1)$ twist of our Landau-Ginzburg model. In the case of Fermat type potential this identification was already given by the author [5]. We try to extend our results to the potentials of other types, i.e. chain and loop potentials.

In [5] it was not possible to identify the $(-1, 1)$ states written in the form $\prod_i {X_i}^{l_i} \left| j^{-l} \right\rangle_{\rm (a,c)}$ with toric data. In this paper, however, we have succeeded in the toric interpretation of these states. Futhermore we can show that these states correspond to the so-called "twisted sector problem" in [2]. Roughly speaking, this problem implies that toric geometry fails to give us the techniques for describing the whole Kähler moduli space and the whole complex structure moduli space. It was pointed out that under certain circumstances we can solve this problem by deforming a polyhedron obtained from a defining equation of a Calabi-Yau manifold [6] . We find that in the Landau-Ginzburg context this construction corresponds to the orbifold construction of [7]. This understanding enables us to propose a useful method to solve the twisted sector problem. Furthermore we will show that the Mori vectors, which are needed to study the moduli space of Calabi-Yau manifolds, can be obtained from the relation of the twist operators of our orbifoldization. This orbifold construction tells us that the moduli spaces of the original Landau-Ginzburg string vacua and their orbifolds are connected. This structure can be shown in terms of our correspondence between Landau-Ginzburg models and toric geometry.

This paper is organized as follows. In section 2 we briefly review the construction of Calabi-Yau manifolds in terms of toric geometry. The twisted sector problem is explained here. The Landau-Ginzburg models and their mirror symmetry are given in section 3. To study mirror symmetry the useful ingredients are the phase symmetries of the superpotentials for the Landau-Ginzburg models. This procedure is explained in this section. In terms of the phase symmetries the correspondence between Landau-Ginzburg models and toric geometry is presented.

Section 4 is devoted to the orbifold construction and its toric aspects. In the previous section we have presented the one-to-one identification between the $(-1, 1)$ states and the toric data. Using this identification, a useful method to resolve the twisted sector problem is proposed. This mehtod is based on the orbifold construction. For Landau-Ginzburg models of our interest, this orbifold construction requires the deformation of the superpotential. This deformation should correspond to the deformation of toric data. As a result the moduli spaces of the original and the orbifoldized theories are connected. We will show the useful formula to obtain the appropriate twist operators for this orbifoldization in terms of the Landau-Ginzburg analysis. The relation between the phase symmetries and the Mori vectors are also discussed. Typical examples are studied in detail.

In section 5 we discuss another application of our toric interpretation of Landau-Ginzburg models in the context of $K3$-fibred Calabi-Yau manifolds and the Seiberg-Witten geometry. An interesting example is given from which we can obtain a quantum field theory at the orbifold point in the moduli space. Finally, in section 6 some speculative remarks are given.

2 General construction of Calabi-Yau hypersurfaces in toric varietes

Let us briefly review a method of toric geometry [1, 2]. In toric geometry,a pair of (Δ, Δ^*) gives a Calabi-Yau manifold, where Δ is a (Newton) polyhedron corresponding to monomials and Δ^* is a dual (or polar) polyhedron describing the resolution of singularities. A point on a one- or two-dimensional face of Δ^* corresponds to a $(1, 1)$ form coming from resolution.

First, we consider the Fermat type quasi-homogeneous polynomial in the weighted

complex projective space $WCP^4_{(w_1,w_2,w_3,w_4,w_5)}[d]$, where w_i are weights and the degree $d=\sum_{i=1}^5 w_i$. The corresponding Calabi-Yau hypersurface consists of monomials z_i^{d/w_i} $(i=1,\cdots,5)$. We assume $w_1=1$, so that the polyhedron obtained from the hypersurface is 4-dimensional. The associated 4-dimensional integral convex polyhedron $\Delta(w)$ is the convex hull of the integral vectors m of the exponents of all quasi-homogeneous monomials of degree d shifted by $(-1,\ldots,-1)$, i.e. $\prod_{i=1}^5 z_i^{m_i+1}$:

$$\Delta(w):=\{(m_1,\ldots,m_5)\in \mathbb{R}^5 | \sum_{i=1}^5 w_i m_i = 0, m_i \geq -1\}. \tag{2.1}$$

This implies that only the origin is the point in the interior of Δ. Its dual polyhedron is defined by

$$\Delta^* = \{\ (x_1,\ldots,x_4)\ |\ \sum_{i=1}^4 x_i y_i \geq -1 \text{ for all } (y_1,\ldots,y_4)\in\Delta\ \}. \tag{2.2}$$

In our case it is known that (Δ,Δ^*) is a reflexive pair. An l-dimensional face $\Theta\subset\Delta$ can be represented by specifying its vertices $\mathrm{v}_{i_1},\cdots,\mathrm{v}_{i_k}$. Then the dual face Θ^* is a $(4-l-1)$-dimensional face of Δ^* and defined by

$$\Theta^* = \{x\in\Delta^*\ |\ (x,\mathrm{v}_{i_1})=\cdots=(x,\mathrm{v}_{i_k})=-1\}, \tag{2.3}$$

where $(*,*)$ is the ordinary inner product.

For models of Fermat type, we then always obtain as vertices of $\Delta(w)$

$$\begin{aligned}\nu_1 &= (-1,-1,-1,-1), \nu_2 = (d/w_2-1,-1,-1,-1),\ \nu_3=(-1,d/w_3-1,-1,-1),\\ \nu_4 &= (-1,-1,d/w_4-1,-1),\ \nu_5=(-1,-1,-1,d/w_5-1),\end{aligned} \tag{2.4}$$

and for the vertices of the dual polyhedron $\Delta^*(w)$ one finds

$$\begin{aligned}\nu_1^* &= (-w_2,-w_3,-w_4,-w_5),\\ \nu_2^* &= (1,0,0,0),\quad \nu_3^*=(0,1,0,0),\quad \nu_4^*=(0,0,1,0),\quad \nu_5^*=(0,0,0,1).\end{aligned} \tag{2.5}$$

Batyrev showed the following useful formula for non-trivial cohomologies [1];

$$h^{1,1}=l(\Delta^*)-5-\sum_{\textbf{codimension }\boldsymbol{\Theta^*=1}} l'(\Theta^*)+\sum_{\textbf{codimension }\boldsymbol{\Theta^*=2,\Theta^*\in\Delta^*}} l'(\Theta^*)l'(\Theta), \tag{2.6}$$

$$h^{2,1} = l(\Delta) - 5 - \sum_{\textbf{codimension } \boldsymbol{\Theta}\mathbf{=1}} l^{'}(\Theta) + \sum_{\textbf{codimension } \boldsymbol{\Theta}\mathbf{=2}, \boldsymbol{\Theta \in \Delta}} l^{'}(\Theta) l^{'}(\Theta^*), \quad (2.7)$$

where $l(\Delta)$ ($l(\Delta^*)$) denotes the number of integral points in Δ (Δ^*). The symbol $l^{'}(\Theta)$ ($l^{'}(\Theta^*)$) denotes the number of integral points in the interior of Θ (Θ^*).

The non-zero contributions of the last terms of (2.6) and (2.7) are called "twisted sector problem" [2] If this problem occurs, toric geometry is not sufficient to give us the techniques for describing the whole Kähler moduli space and the whole complex structure moduli space. In section 4, we will explain this problem more in depth and propose a method to circumvent this undesirable situation.

For the non-Fermat polynomials which always contain the another type of monomials ${z_i}^{a_i} z_j$, it was shown [6, 8] that toric geometry works in a similar manner. For example, if the defining equation contains the monomial ${z_4}^{a_4} z_5$ in stead of z_4^{d/w_4}, we have to consider the vertex $(-1, -1, a_4 - 1, 0)$ instead of ν_4 in (2.4).

The defining equation for a mirror manifold is obtained as the zero locus of the Laurent polynomial

$$p^{'} = \sum_i a_i {y_i}^{{\nu_i}^*} \quad (2.8)$$

where y_i are coordinates of a canonical torus $(\mathrm{C}^*)^4$ and the coefficients a_i are parameters characterizing the complex structure of the mirror manifold. If one wants to get the hypersurface as a quasihomogeneous polynomial, one should specify an étale map [6];

$$y_i = \frac{\bar{m}_i}{\bar{X}_1 \bar{X}_2 \bar{X}_3 \bar{X}_4 \bar{X}_5}, \quad (2.9)$$

where $\bar{X}_i$ are the fields and $\bar{m}_i$ is the appropriate monomial of the transposed polynomial [9] which will be explained in section 3.1 in the context of the Landau-Ginzburg superpotentials.

3 Toric geometrical aspects of Landau-Ginzburg string vacua

3.1 Landau-Ginzburg models and their mirrors

A Landau-Ginzburg model is characterized by a superpotential $W(X_i)$ where X_i are $N = 2$ chiral superfields. The Landau-Ginzburg orbifolds [3] are obtained by

quotienting with an Abelian symmetry group G of $W(X_i)$, whose element g acts as an $N \times N$ diagonal matrix, $g : X_i \rightarrow e^{2\pi i \tilde{\theta}_i^{\,g}} X_i$. Here $0 \leq \tilde{\theta}_i^{\,g} < 1$. Of course the $U(1)$ twist $j : X_i \rightarrow e^{2\pi i q_i} X_i$ generates the symmetry group of $W(X_i)$, where $q_i = \frac{w_i}{d}$, $W(\lambda^{w_i} X_i) = \lambda^d W(X_i)$ and $\lambda \in \mathbb{C}^*$. In this paper, we further require that $w_1 = 1$ since the toric description of the corresponding Calabi-Yau manifolds and their mirror manifolds are well-known [1, 2]. Using the results of Intriligator and Vafa [3], we can construct the (c, c) and (a, c) rings, where c (a) denotes chiral (anti-ciral). Also we could have the left and right $U(1)$ charges of the ground state $|h\rangle_{\rm (a,c)}$ in the h-twisted sector of the (a, c) ring. In terms of spectral flow, $|h\rangle_{\rm (a,c)}$ is mapped to the (c,c) state $|h'\rangle_{\rm (c,c)}$ with $h' = hj^{-1}$. Then the charges of the (a,c) ground state of h-twisted sector $|h\rangle_{\rm (a,c)}$ are obtained to be

$$\begin{pmatrix} J_0 \\ \bar{J}_0 \end{pmatrix} \quad |h\rangle_{\rm (a,c)} = \begin{pmatrix} -\sum_{\tilde{\theta}_i^{\,h'}>0} (1 - q_i - \tilde{\theta}_i^{\,h'}) + \sum_{\tilde{\theta}_i^{\,h'}=0} (2q_i - 1) \\ \sum_{\tilde{\theta}_i^{\,h'}>0} (1 - q_i - \tilde{\theta}_i^{\,h'}) \end{pmatrix} \quad |h\rangle_{\rm (a,c)}. \tag{3.1}$$

First we consider the Landau-Ginzburg models with the Fermat type superpotentials

$$W_{Fermat} = X_1^{a_1} + X_2^{a_2} + \cdots + X_n^{a_n}, \tag{3.2}$$

and their (a, c) states. We define the phase symmetries ρ_i which act on X_i as

$$\rho_i X_i = e^{-2\pi i q_i} X_i, \tag{3.3}$$

with trivial action for other fields. The operator ρ_i can be represented by a diagonal matrix whose diagonal matrix elements are 1 except for $(\rho_i)_{i,i} = e^{-2\pi i q_i}$. It is obvious that

$$j^{-1} = \rho_1 \cdots \rho_5. \tag{3.4}$$

In ref.[13] the mirror map for the (a, c) ground states in the j^{-l}-twisted sector $\left|j^{-l}\right\rangle_{\rm (a,c)}$ are considered. In the j^{-l}-twisted sector, if a field X_i is invariant then

$$\rho_i^{-l} = \rho_i^{-l_i} = \text{identity}, \tag{3.5}$$

where $-l_i \equiv -l \bmod a_i$ and one gets

$$j^{-l} = \prod_{-lq_i \notin \mathbb{Z}} \rho_i^{l_i}. \tag{3.6}$$

So, we may represent $\left|j^{-l}\right\rangle_{(\mathrm{a,c})} = \left|\prod_{-lq_i\notin\mathbb{Z}} \rho_i^{l_i}\right\rangle_{(\mathrm{a,c})}$. Furthermore, we can calculate the $U(1)$ charges of this ground state and the result is

$$(-\sum_{-lq_i\notin\mathbb{Z}} l_i q_i, \sum_{-lq_i\notin\mathbb{Z}} l_i q_i). \tag{3.7}$$

Using the phase symmmetry ρ_i considered above, one can obtain the following mirror map for Landau-Ginzburg model [13];

$$\left|\prod_{-lq_i\notin\mathbb{Z}} \rho_i^{l_i}\right\rangle_{(\mathrm{a,c})} \quad \stackrel{\text{mirror partner}}{\longleftrightarrow} \quad \prod_{-lq_i\notin\mathbb{Z}} X_i^{l_i}. \tag{3.8}$$

There are potentials of other type, i.e. so-called chain (tadpole) potential

$$W_{chain} = X_1^{a_1} X_2 + \cdots + X_{n-1}^{a_{n-1}-1} X_n + X_n^{a_n} \tag{3.9}$$

and loop potential

$$W_{loop} = X_1^{a_1} X_2 + \cdots + X_n^{a_n} X_1. \tag{3.10}$$

In general the Landau-Ginzburg superpotentials contain Fermat, chain and loop terms. To study the mirror theory, we have to consider the transposed potentials [9]. For the chain potentials, their transposed potentials are

$$\bar{W}_{chain} = \bar{X}_1{}^{a_1} + \bar{X}_1 \bar{X}_2{}^{a_2} + \cdots + \bar{X}_{n-1} \bar{X}_n{}^{a_n}, \tag{3.11}$$

and for the loop potentials

$$\bar{W}_{loop} = \bar{X}_1 \bar{X}_2{}^{a_1} + \cdots + \bar{X}_{n-1} \bar{X}_n{}^{a_{n-1}} + \bar{X}_n \bar{X}_1{}^{a_n}. \tag{3.12}$$

Their corresponding defining equations are the hypersurfaces of the mirror manifold. For the Fermat type potentials the transposed potentials are the same form as the original ones.

The phase symmetreis ρ_i of chain and loop potentials were studied by Kreuzer [11]. Let us briefly summarize the results. The definition of phase transformations is

$$\rho_i X_l = \exp(2\pi \mathrm{i} \varphi_l^{(i)}) X_l. \tag{3.13}$$

For the chain potentials, we have

$$\varphi_l^{(i)} = \frac{(-1)^{l-i+1}}{a_i \cdots a_l} \quad \text{for } 1 \leq l \leq i, \tag{3.14}$$

and $\varphi^{(i)}_{i+1} = \cdots = \varphi^{(i)}_n = 0$. Moreover we have the following relations,

$$a_l \varphi^{(i)}_l + \varphi^{(i)}_{l+1} = -\delta^i_l, \tag{3.15}$$

$$a_{i+1} \varphi^{(i+1)}_l + \varphi^{(i)}_l = -\delta^{i+1}_l. \tag{3.16}$$

From these relations, we can see

$$(\rho_i)^{a_i} \rho_{i-1} = 1, \tag{3.17}$$

$$q_i = -\sum_{j=i}^{n} \varphi^{(j)}_i. \tag{3.18}$$

For the loop potentials, one obtains

$$\varphi^{(i)}_{i+l} = \frac{(-1)^{n-l} a_i \cdots a_{i+l-1}}{\Gamma} \quad \text{for } 0 \le l < n, \tag{3.19}$$

where $\Gamma = A - (-1)^n, \;\; A = a_1 a_2 \cdots a_n$. In a similar fashion as chain potentials, one can see

$$a_l \varphi^{(i)}_l + \varphi^{(i)}_{l+1} = -\delta^i_{l+1}, \tag{3.20}$$

$$a_i \varphi^{(i+1)}_l + \varphi^{(i)}_l = -\delta^i_l \tag{3.21}$$

and

$$\rho_i (\rho_{i+1})^{a_i} = 1, \tag{3.22}$$

$$q_i = -\sum_{l=1}^{n} \varphi^{(l)}_i. \tag{3.23}$$

For both cases we may represent any twist h uniquely in the form

$$h = \prod \rho_i^{\alpha_i} \tag{3.24}$$

where $0 \le \alpha_i < a_i$. A simple calculation shows that the $U(1)$ charge of the state $|h\rangle_{(\mathrm{a,c})}$ is

$$(-\sum_{\text{all } i} \varphi_i, \sum_{\text{all } i} \varphi_i) \tag{3.25}$$

where $\varphi_i = \sum_{l=i}^{n} \alpha_l \varphi^{(l)}_i$.

Due to the obsevation of the $U(1)$ charge, one can obtain the following mirror map for Landau-Ginzburg model [11];

$$\Big|\prod \rho_i^{l_i}\Big\rangle_{(\mathrm{a,c})} \quad \overset{\text{mirror partner}}{\longleftrightarrow} \quad \prod \bar{X}_i^{l_i}, \tag{3.26}$$

where $\bar{X}_i$ are the fields of the transposed potential. The special class of this mirror symmetry, namely

$$(-1,1) \text{ states } \left|j^{-l}\right\rangle_{(\mathrm{a,c})} \quad \overset{\text{mirror partner}}{\longleftrightarrow} \quad (1,1) \text{ states } \prod_{\varphi_i \notin \mathbb{Z}} \bar{X}_i^{l_i} |0\rangle_{(\mathrm{a,c})}, \tag{3.27}$$

is important since the monomials $\prod_i \bar{X}_i^{l_i}$ correspond to the monomial deformations of the complex structure of the mirror manifold.

3.2 Correspondence between Landau-Ginzburg model and toric geometry

In this section,we study the toric geometrical structure of Landau-Ginzburg model. We will show that the Batyrev's formula of eqs.(2.6) and (2.7) exactly count the number of the $(-1,1)$ and $(1,1)$ states, respectively, as expected. This analysis enables us to tell that which states sould correspond to the twisted sector problem.

First, we consider the Fermat type potentials and study their toric aspects. This was already done in [5], so we briefly review the discussion of [5] and make it more precise. Then the generalization to the chain and loop potentials will be given.

The eq.(3.6) is the key equation for our purpose. This implies

$$-lq_i = u_i - l_i q_i \qquad \text{for } i = 1 \sim 5, \tag{3.28}$$

where $-l_i$ are defined to be $-a_i + 1 \leq -l_i \leq 0$. Thus u_i are uniquely determined. Clearly, u_i are integers and $l_i = 0$ if X_i is invariant under j^{-l} action. u_1 always vanish since $w_1 = 1$.

Let $u \equiv (u_2, u_3, u_4, u_5)$ be an integral vector. It can be shown [5] that u is on a face of a dual polyhedron $\Delta^*(w)$. So we should assert that u is just an integral point inside $\Delta^*(w)$, which can be identified with the exceptional divisors. Through this identification, we obtain the one-to-one correspondence between the $(-1,1)$ state $\left|j^{-l}\right\rangle_{(\mathrm{a,c})}$ and the exceptional divisor. Moreover, once this identification is made, we can obtain the monomial-divisor mirror map [12] for Calabi-Yau manifolds.

A simple observation shows that the above identification is equivalent to the following identification

$$\rho_i^{a_i} = 1 \quad \rightarrow \quad (0, \cdots, 1, \cdots, 0) \in \Delta^*, \tag{3.29}$$

where the components of the vectors are zero except for the $i-$th entry being 1. More concretely, for the twist j^{-l} eq.(3.28) implies

$$\rho_i^l = (\rho_i^{a_i})^{-u_i} \rho_i^{l_i}. \tag{3.30}$$

If the $(-1,1)$ state $\left| j^{-l} \right\rangle_{(\mathrm{a,c})}$ with 2 and 3 invariant field under j^{-l} action, the toric data u obtained by the above method lies on the 2 and 1 dimensional face Θ^*, respectively. If the $(-1,1)$ state $\left| j^{-l} \right\rangle_{(\mathrm{a,c})}$ has only one invariant field, we can obtain the toric data $\nu^{*(\alpha)}$ by the same mehtod and $\nu^{*(\alpha)}$ lies on a codimension 1 (i.e. 3-dimensional) face Θ^*. Moreover, in this case we can further obtain the new toric data $\nu^{*(\beta)}$ from the descendent twist j^{-l+1}. If only one field is invariant under j^{-l+1} action, we can obtain the third toric data $\nu^{*(\gamma)}$ from the twist j^{-l+2}. This successive process will stop when the twist j^{-l+n} has 2 or 3 invariant fields. These results are in complete agreement with the subtraction of $\sum_{\textbf{codimension } \boldsymbol{\Theta}^*\mathbf{=1}} l'(\Theta^*)$ in the Batyrev's formula (2.6).

It was noticed in [5] that the mirror partners of the $(-1,1)$ states written in the form $\prod_{\tilde{\theta}_i^{h'}=0} X_i^{l_i} |h\rangle_{(\mathrm{a,c})}$ can not be described by the monomials of fields in the transposed potential. So it is natural to expect that these states correspond to the twisted sector problem. To prove this, we should note the fact that the number of $(-1,1)$ states $\left| j^{-l} \right\rangle_{(\mathrm{a,c})}$ with two invariant fields (say, X_i and X_j) is $l'(\Theta^*)$, where Θ^* is the 2-dimensional dual face of 1-dimensional face $\Theta(\nu_i, \nu_j)$. The toric data ν_i is the vector whose components are -1 except for the $i-$th entry being $a_i - 1$. This vector corresponds to the monomial $X_i^{a_i}$. If $\Theta(\nu_i, \nu_j)$ contains integral points in its interior, these points should correspond to the monomials $X_i^{\alpha_i} X_j^{\alpha_j}$, and the number of such monomials is shown to be $l'(\Theta)$. Since the $U(1)$ charge of the state $\left| j^{-l+1} \right\rangle_{(\mathrm{a,c})}$ with two invariant fields is $(-2+\sum_{lq_i \in \mathbb{Z}} q_i, \sum_{lq_i \in \mathbb{Z}} q_i)$ and $\alpha_i q_i + \alpha_j q_j = 1$, the states $X_i^{\alpha_i - 1} X_j^{\alpha_j - 1} \left| j^{-l+1} \right\rangle_{(\mathrm{a,c})}$ are $(-1,1)$ states. One can see that the number of such states is $l'(\Theta) l'(\Theta^*)$ with 2-dimensional dual face Θ^*.

To generalize the above result to chain and loop potentials, we should extend the identification of (3.30) to

$$\rho_i^{a_i} = 1 \quad \rightarrow \quad (0, \cdots, 1, \cdots, 0) \in \Delta^* \tag{3.31}$$

and

$$\rho_i^{a_i} \rho_j = 1 \quad \rightarrow \quad (0, \cdots, 1, \cdots, 0) \in \Delta^*, \tag{3.32}$$

where the components of the vectors are zero except for the $i-$th entry being 1 as before. It can be shown that these identifications coincide with the equivalence between the mirror map of (3.26) and the étale map of eq.(2.9).

Furthermore we can point out the relation between the Mori vectors and the phase symmetries. A Mori cone which consists of Mori vectors is a dual cone of a Kähler cone and can describe the good coordinates for a moduli space of complex structure [2]. A Mori vector represents a relation of the monomials in a defining equation of a mirror Calabi-Yau manifold. Since the mirror map (3.27) connects these monomials and the j^{-l} twist, we can expect that a Mori vector translates into a relation of j^{-l} or ${\rho_i}^{l_i}$ twists. Indeed, a following type of relation,

$$\prod_i \rho_i^{l_i} = \text{identity}, \tag{3.33}$$

leads to a Mori vector. In the later section we will show some examples.

4 Mirror symmetry and the web of string vacua

4.1 Twisted sector problem revisited

It is a well known fact [1, 2] that in certain situations not all of the Kähler moduli space can be described by toric divisors of the Calabi-Yau manifold $\mathcal{M}$. Similarly, the deformation of the defining equation of the hypersurface of $\mathcal{M}$ in a toric variety is in general not sufficient to describe the complex structure moduli space of $\mathcal{M}$. More precisely, as remarked under (2.6) and (2.7), there are non toric divisors in $\mathcal{M}$ when

$$\sum_{\textbf{codimension } \boldsymbol{\Theta^*=2}, \boldsymbol{\Theta^* \in \Delta^*}} l^{'}(\Theta^*) l^{'}(\Theta) \tag{4.1}$$

is non-zero. In a similar fashion, when

$$\sum_{\textbf{codimension } \boldsymbol{\Theta=2}, \boldsymbol{\Theta \in \Delta}} l^{'}(\Theta) l^{'}(\Theta^*) \tag{4.2}$$

is non-zero there are non-algebraic deformations of the complex structure deformations of the complex structure of $\mathcal{M}$.

In section 3.2, we have shown that in the Landau-Ginzburg context these numbers (4.1) ,(4.2) are equivalent to the number of the states $\prod_{\tilde{\theta}_i{}^{h'} \in \mathbb{Z}} X_i^{l_i} \left| j^{-l} \right\rangle_{(\mathrm{a,c})}$ with $U(1)$ charge $(-1, 1)$, $(1, 1)$ respectively.

4.2 Orbifold construction and the web of Landau-Ginzburg vacua

In this section, we would like to propose a method which circumvent this twisted sector problem under some circumstances. As a result either all of the non-toric divisors in (4.1) or the non-algebraic deformations of the complex structure in (4.2), can be treated by this method. In the following we will concentrate ourselves on the Kähler moduli space and the corresponding $(-1,1)$ states.

The main idea is orbifold construction, i.e. we further divide the original theory by a new discrete symmetry, whose generator is denoted by g, and project out the unexpected states $\prod_{\tilde{\theta}_i{}^{h'} \in \mathbb{Z}} X_i^{l_i} \left| j^{-l} \right\rangle_{(\mathrm{a,c})}$ with $U(1)$ charge $(-1,1)$. At the same time, we have new twisted states $\left| g^{-m} j^{-l'} \right\rangle_{(\mathrm{a,c})}$ with charge $(-1,1)$, which can be identified with toric data. According to this orbifoldization, the superpotential should be deformed, as discussed in [7]

On the toric geometry side, after this procedure is done we have new polyhedra $\tilde{\Delta}^*$ and $\tilde{\Delta}$ such that

$$\tilde{\Delta}^* \supset \Delta^*, \quad \tilde{\Delta} \subset \Delta. \tag{4.3}$$

This relation implies that the moduli spaces of the original theory and the orbifoldized thory are connected [8, 10].

Can we find such a suitable twist? In the following, we will show that under some circumstances we can find the appropriate g for our construction.

Consider the state $\prod_{\tilde{\theta}_i{}^{h'}=0} X_i^{\alpha_i} \left| j^{-l+1} \right\rangle_{(\mathrm{a,c})}$ where $0 \leq \alpha_i \leq a_i - 2$. The $U(1)$ charge of this state is obtained to be

$$\begin{pmatrix} -\sum_{\tilde{\theta}_i{}^{h'}>0} (1 - q_i - \tilde{\theta}_i{}^{h'}) + \sum_{\tilde{\theta}_i{}^{h'}=0} (2q_i - 1) + \sum_{\tilde{\theta}_i{}^{h'}=0} \alpha_i q_i \\ \sum_{\tilde{\theta}_i{}^{h'}>0} (1 - q_i - \tilde{\theta}_i{}^{h'}) + \sum_{\tilde{\theta}_i{}^{h'}=0} \alpha_i q_i \end{pmatrix}, \tag{4.4}$$

where $h = j^{-l}$ and $h' = hj^{-1}$.

In this case the appropriate g is found to be

$$g^{-1} = \prod_{\tilde{\theta}_i{}^{h'}=0} \rho_i{}^{\alpha_i+1}. \tag{4.5}$$

Using this g, the new twisted state $\left| g^{-1} j^{-l+1} \right\rangle_{(\mathrm{a,c})}$ will appear and its $U(1)$ charge is obtained to be $(-Q, Q)$, where $Q = \sum_{\tilde{\theta}_i{}^{h'}>0} (1 - q_i - \tilde{\theta}_i{}^{h'}) + \sum_{\tilde{\theta}_i{}^{h'}=0} \alpha_i q_i$.

We see that if

$$\sum_{\tilde{\theta}_i{}^{h'}=0} (2q_i - 1) + \sum_{\tilde{\theta}_i{}^{h'}=0} \alpha_i q_i = - \sum_{\tilde{\theta}_i{}^{h'}=0} \alpha_i q_i, \tag{4.6}$$

then the two states $\prod_{\tilde{\theta}_i{}^{h'}=0} X_i{}^{\alpha_i} \left| j^{-l+1} \right\rangle_{(\mathrm{a,c})}$ and $\left| g^{-1} j^{-l+1} \right\rangle_{(\mathrm{a,c})}$ have same $U(1)$ charge. To obtain the toric data of the orbifoldized theory, we should use the identifications of (3.31) and (3.32) in j^{-l+1}.

4.3 Examples

As a first example, we consider the Landau-Ginzburg model with the superpotential

$$W_1 = X_1^{12} + X_2^{12} + X_3^{12} + X_4^4 + X_5^2, \tag{4.7}$$

with $U(1)$ charges of X_i being

$$\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{4}, \frac{1}{2} \right), \tag{4.8}$$

which corresponds to the hypersurface in $WCP^4_{(1,1,1,3,6)}$[12] and were studied in ref.[6].

We find that there are three $(-1, 1)$ states $|j^{-2}\rangle_{(\mathrm{a,c})}$, $|j^{-4}\rangle_{(\mathrm{a,c})}$ and $X_4|j^{-3}\rangle_{(\mathrm{a,c})}$. From the first two of these states we can easily obtain the toric data and the results are displayed in Table 1, where Dim. means the dimension of the dual face on which the point ν^* lies. The reader should remind that for Calabi-Yau threefolds, a point lying on a face of Dim. = 1 or 2 corresponds to a toric divisors and $\nu_0^* = (0, 0, 0, 0)$ corresponds to the canonical divisor. Note that since only one field is invariant under the twist j^{-2}, we should consider the descendent twist j^{-1} to obtain the toric data completely, as explained.

The remaining $(-1, 1)$ state represented by $X_4|j^{-3}\rangle_{(\mathrm{a,c})}$ should correspond to so-called twisted sector problem in toric geometry. divisors. To solve this twisted sector problem, we will use the mehtod proposed in the section 4.2. Let us remind that the essential ingredient is an orbifold construction. If one finds an appropriate orbifoldization, the $(-1, 1)$ states such as $X_4|j^{-3}\rangle_{(\mathrm{a,c})}$ are projected out and new $(-1, 1)$ states will appear in the new twisted sectors. For our example, we can find such an appropriate orbifoldization.

(a,c) state	twist	ν_j^* vector	Dim.
$\lvert j^{-2}\rangle_{(\mathrm{a,c})}$	j^{-2}	$\nu_6^* = (0,0,0,-1)$	3
	j^{-1}	$\nu_0^* = (0,0,0,0)$	4
$\lvert j^{-4}\rangle_{(\mathrm{a,c})}$	j^{-4}	$\nu_7^* = (0,0,-1,-2)$	2

Table 1: The toric data of the $(-1,1)$ states in the Landau-Ginzburg orbifold of W_1

Consider the following superpotential

$$W_2 = Y_1^{12} + Y_2^{12} + Y_3^{12} + Y_4^2 Y_5 + Y_5^2. \tag{4.9}$$

Note that the set of fields Y_i have the same $U(1)$ charges as X_i's. So W_2-theory has the same topological numbers as W_1-theory. Hence one may identify the fields X_i and Y_i;

$$X_i = Y_i \ \text{ for } \ i = 1,2,3, \quad X_4^4 = Y_5^2, \ \ X_5^2 = Y_4^2 Y_5. \tag{4.10}$$

For one-to-one identification, we should further identify

$$X_4 \sim -X_4, \quad X_5 \sim -X_5. \tag{4.11}$$

Since this twist is not included in the $U(1)$ twist j^{-l}, W_2-theoy can be considered as $\mathbb{Z}_2$ orbifold of W_1-theory [7]. We denote by g this $\mathbb{Z}_2$ twist. From the results of section 4.2, we find the appropriate g of (4.5) is,

$$g^{-1} = {\rho_4}^2 \rho_5, \tag{4.12}$$

this agrees with $\mathbb{Z}_2$ identification (4.11). The new $(-1,1)$ state is $\lvert g^{-1} j^{-3}\rangle_{(\mathrm{a,c})}$. Below one finds our results. The twisted sector problem has resolved as expected.

Our results shows that the toric data (Δ_i, Δ_i^*) which correspond to W_i-theory satisfy

$$\Delta_1 \supset \Delta_2, \ \text{ and } \ \Delta_1^* \subset \Delta_2^*. \tag{4.13}$$

These properties of toric data imply that the mouli spaces of these two theories are connected [8, 10]. From our construction, the singular $\mathbb{Z}_2$ orbifold point should be considered that the superpotential is W_2 and $(-1,1)$ states are $\lvert j^{-2}\rangle_{(\mathrm{a,c})}$ and

(a,c) state	twists	ν_j^* vector	Dim.
$\vert j^{-2}\rangle_{(\mathrm{a,c})}$	j^{-2}	$\nu_6^* = (0,0,0,-1)$	3
	j^{-1}	$\nu_0^* = (0,0,0,0)$	4
$\vert j^{-4}\rangle_{(\mathrm{a,c})}$	j^{-4}	$\nu_7^* = (0,0,-1,-2)$	2
$\vert g^{-1}j^{-3}\rangle_{(\mathrm{a,c})}$	$g^{-1}j^{-3}$	$\nu_{11}^* = (0,0,-1,-1)$	3
	$g^{-1}j^{-2}$	$\nu_{10}^* = (0,0,-1,0)$	3
	$g^{-1}j^{-1}$	$\nu_9^* = (0,0,-1,1)$	3
	g^{-1}	$\nu_8^* = (0,0,0,1)$	1

Table 2: The toric data of the $(-1,1)$ states in the Landau-Ginzburg orbifold of W_2

$|j^{-4}\rangle_{(\mathrm{a,c})}$. Thus on the singular orbifold point, the corresponding toric data are (Δ_2, Δ_1^*). This is not a dual pair so that one could not obtain a smooth Calabi-Yau manifold. However, this is a boundary point which connects the moduli spaces of the two theories described by the superpotentials W_1 and W_2. Our orbifold construction corresponds to the toric construction in refs.[6, 10], in which the structures of moduli spaces of Calabi-Yau manifolds are studied.

Then, we consider the relation between Mori vectors and the twist operators. In the original theory whose potential is W_1, there is a $(-1,1)$ state $|j^{-4}\rangle_{(\mathrm{a,c})}$ whose toric data is ν_7^*. In terms of the mirror map for Landau-Ginzburg model (3.27), the relation $(j^{-4})^3 = j^{-12} =$ identity translates into $({X_1}^4{X_2}^4{X_3}^4)^3 = ({X_1}^{12})({X_2}^{12})({X_3}^{12})$. This implies the Mori vector $l_1^{(1)} = (0;1,1,1,0,0,-3,0)$. In a similar manner, $j^{-4} = (j^{-1})^4$ translates into $(X_1^4X_2^4X_3^4)(X_4^4)(X_5^2)^2 = (X_1X_2X_3X_4X_5)^4$ and we get the Mori vector $l_1^{(2)} = (-4;0,0,0,1,2,1,0)$.

In the orbifoldized theory whose potential is W_2, we have a new operator set $\{\tilde{\rho}_i\}$. From the eq.(3.17), we have

$$\tilde{\rho}_1{}^{12} = \tilde{\rho}_2{}^{12} = \tilde{\rho}_3{}^{12} = \tilde{\rho}_4{}^2 = \tilde{\rho}_4\tilde{\rho}_5{}^2 = \text{identity}. \tag{4.14}$$

Note that from these basic relations of the operators $\tilde{\rho}_i$, we have the new relation $(\tilde{\rho}_4\tilde{\rho}_5^2)^2(\tilde{\rho}_4{}^2)^{-1} = \tilde{\rho}_5^4 =$ identity. This relation $(\tilde{\rho}_4\tilde{\rho}_5^2)^2(\tilde{\rho}_4{}^2)^{-1} = \tilde{\rho}_5^4 =$ identity gives us the new Mori vector. In terms of the mirror map for Landau-Ginzburg model (3.27), we can obtain $(\bar{Y}_4\bar{Y}_5{}^2)^2 = (\bar{Y}_4{}^2)(\bar{Y}_5{}^4)$. The corresponding Mori vector is $l_2^{(1)} =$

$(0;0,0,0,1,-2,0,1)$. In this orbifold model, $j^{-4}=(j^{-1})^4$ translates into the Mori vector $l_2^{(2)}=(-4;0,0,0,0,4,1,-1)$. As a result, the Mori vectors satisfy the relation $l_1^{(2)}=l_2^{(1)}+l_2^{(2)}$ [6]. We can understand this relation as the change of the set of twist operators $\{\rho_i\}$ of the phase symmetry, or equivalently, as the orbifoldization.

As a second example, we consider the hypersurface in $WCP_{(1,2,3,6,6)}[18]$ which was considered in [15, 5]. In this case the orbifoldized potential contains a loop part. The defining superpotential is

$$W_3=X_1^{18}+X_2^9+X_3^6+X_4^3+X_5^3. \tag{4.15}$$

There are seven $(-1,1)$ states and two of them correspond to the twisted sector problem, i.e. $X_4|j^{-2}\rangle_{(\mathrm{a,c})}$ and $X_5|j^{-2}\rangle_{(\mathrm{a,c})}$. The appropriate twist is

$$g^{-1}={\rho_4}^2\rho_5, \tag{4.16}$$

and this generates the $\mathbb{Z}_3$ being

$$X_4\sim\alpha^2X_4,\quad X_5\sim\alpha X_5 \tag{4.17}$$

where α is a cubic loot of unity. After the orbifoldization, the deformed superpotential includes a loop potential and obtained to be

$$W_4=Y_1^{18}+Y_2^9+Y_3^6+Y_4^2Y_5+Y_4Y_5^2. \tag{4.18}$$

In the orbifoldized theory, the new $(-1,1)$ states are $|g^{-1}j^{-2}\rangle_{(\mathrm{a,c})}$ and $|g^{-2}j^{-2}\rangle_{(\mathrm{a,c})}$. The corresponding toric data are displayed in Table 3 with the descendent twists. Our results show that the twisted sector problem has circumvented as expected.

Our third example [8] is the hypersurface in $WCP_{(1,2,2,2,7)}[14]$. The corresponding superpotential is

$$W_5=X_1^{14}+X_2^7+X_3^7+X_4^7+X_5^2, \tag{4.19}$$

and there are two $(-1,1)$ states $|j^{-2}\rangle_{(\mathrm{a,c})}$ and $|j^{-8}\rangle_{(\mathrm{a,c})}$. Once we identify

$$X_1\sim-X_1,\quad X_5\sim-X_5, \tag{4.20}$$

then we obtain the orbifoldized potential

$$W_6=Y_1^7+Y_2^7+Y_3^7+Y_4^7+Y_1Y_5^2. \tag{4.21}$$

(a,c) state	twists	ν_j^* vector	Dim.
$\vert g^{-1}j^{-2}\rangle_{(\mathrm{a,c})}$	$g^{-1}j^{-2}$	$\nu_6^* = (0,0,-1,0)$	3
	$g^{-1}j^{-1}$	$\nu_8^* = (0,0,1,-1)$	3
	g^{-1}	$\nu_{10}^* = (0,0,1,0)$	1
$\vert g^{-2}j^{-2}\rangle_{(\mathrm{a,c})}$	$g^{-2}j^{-2}$	$\nu_7^* = (0,0,0,-1)$	3
	$g^{-2}j^{-1}$	$\nu_9^* = (0,0,-1,1)$	3
	g^{-1}	$\nu_{11}^* = (0,0,0,1)$	1

Table 3: The toric data of Landau-Ginzburg orbifold of W_4

The corresponding hypersurface is in $WCP_{(1,1,1,1,3)}$[7]. This theory includes two $(-1,1)$ states $|j^{-1}\rangle_{(\mathrm{a,c})}$ and $|j^{-2}\rangle_{(\mathrm{a,c})}$. Though their weights are diferent, these two theories should be equivalent since the identification (4.20) is a part of projective equivalnce of $WCP_{(1,2,2,2,7)}$[14],

$$(X_1, X_2, X_3, X_4, X_5) \sim (\lambda X_1, \lambda^2 X_2, \lambda^2 X_3, \lambda^2 X_4, \lambda^7 X_5), \tag{4.22}$$

with $\lambda = -1$.

The vertices of the dual polyhedron Δ_5^* corresponding to the W_5-theory are

$$\begin{array}{ccc} (-2,-2,-2,-7) & , & (1,0,0,0), \quad (0,1,0,0), \\ (0,0,1,0) & , & (0,0,0,1), \quad (0,0,0,-1), \\ (-1,-1,-1,-3) & , & (-1,-1,-1,-4). \end{array}$$

The toric data $(0,0,0,-1)$ and $(-1,-1,-1,-4)$ correspond to the two $(-1,1)$ states $|j^{-2}\rangle_{(\mathrm{a,c})}$ and $|j^{-8}\rangle_{(\mathrm{a,c})}$, respectively. The toric data $(-1,-1,-1,-3)$ can be identified with the descendent twist j^{-7}.

Turning our attention to the W_6-theory, the vertices of Δ_6^* are

$$\begin{array}{ccc} (-1,-1,-1,-3) & , & (1,0,0,0), \quad (0,1,0,0), \\ (0,0,1,0) & , & (0,0,0,1). \end{array}$$

At first sight, the sets of toric data for these two theories are different. However, we can show that the toric data which are obtained from W_6-theory are the same as the ones from W_5-thory. The results are displayed in Table 4

(a,c) state	twists	ν_j^* vector	Dim.
$\vert j^{-2}\rangle_{(\mathrm{a,c})}$	j^{-2}	$\nu_6^* = (0,0,0,-1)$	3
	j^{-1}	$\nu_7^* = (-1,-1,-1,-4)$	3
	j^{0}	$\nu_8^* = (-2,-2,-2,-7)$	1

Table 4: The toric data of the $(-1,1)$ states in the Landau-Ginzburg orbifold of W_6

As a final example, we consider the hypersurface in $WCP_{(1,1,1,2,5)}[10]^1_{-288}$, where the superscript denotes the number of $(1,1)$ forms and the subscript implies the Euler number. The corresponding superpotential is

$$W_7 = X_1^{10} + X_2^{10} + X_3^{10} + X_4^5 + X_5^2. \tag{4.23}$$

This theory contains only one $(-1,1)$ state $|j^{-1}\rangle_{(\mathrm{a,c})}$ whose toric data is the origin $(0,0,0,0)$ in the dual polyhedron Δ^*. The identification

$$X_3 \sim -X_3, \quad X_5 \sim -X_5, \tag{4.24}$$

leads to the orbifoldized theory whose potential is

$$W_8 = Y_1^{10} + Y_2^{10} + Y_3^5 + Y_4^5 + Y_3 Y_5^2, \tag{4.25}$$

which corresponds to the hypersurface in $WCP_{(1,1,2,2,4)}[10]^3_{-192}$. This theory contains the additional two $(-1,1)$ states, i.e. $|j^{-2}\rangle_{(\mathrm{a,c})}$ and $|j^{-5}\rangle_{(\mathrm{a,c})}$ whose toric data are $(0,0,0,-1)$ and $(0,-1,-1,-2)$, respectively. Our results implies that

$$\Delta_7 \supset \Delta_8, \quad \text{and} \quad \Delta_7^* \subset \Delta_8^*. \tag{4.26}$$

These relations show that the moduli spaces of these two theories are connected, even though their numbers of states, namely their mass spectra, are different. This should be considered as an example of the extremal transition between the topologically different Calabi-Yau manifolds.

5 Application to type II strings

In the context of string duality, the $K3$-fibred Calabi-Yau manifolds are actvily studied. The conjectured duality [14] is that the heterotic string on $K3 \times T^2$ is dual

to the type II string on the Calabi-Yau manifold which admits $K3$-fibration. Such type II strings are the promising candidates to obtain the Seiberg-Witten gauge theory [16] by taking the suitable low energy limit [17, 18, 19].

In this section, we consider the simplest $K3$-fibred Calabi-Yau manifolds constructed in [20], i.e. the Fermat type Calabi-Yau manifolds in $WCP_{(1,1,2w_3,2w_4,2w_5)}[2d]$ whose $K3$-fibre are $WCP_{(1,w_3,w_4,w_5)}[d]$, where $d = 1+w_3+w_4+w_5$. For these models, our Landau-Ginzburg analysis shows that the $(-1,1)$ state $\left|j^{-d}\right\rangle_{(\mathrm{a,c})}$ always exists and that this state corresponds to the toric data $\nu_6^* = (0,-w_3,-w_4,-w_5)$. Since the dual polyhedron Δ^* contains the two vertices $\nu_1^* = (-1,-2w_3,-2w_4,-2w_5)$ and $\nu_2^* = (1,0,0,0)$, we see $\nu_6^* = \frac{1}{2}(\nu_1^* + \nu_2^*)$. This corresponds to the relation $(j^{-d})^2 = j^{-2d}$. In other words, ν_1^*, ν_2^* and ν_6^* correspond to the monomials X_1^{2d}, X_2^{2d} and $X_1^d X_2^d$, respectively, of the defining equation of the mirror manifold on which a type IIB string gets compactified. The existence of these monomials is necessary to obtain the Seiberg-Witten geometry [17, 18, 19]. More precisely, the defining equation of the $K3$-fibred Calabi-Yau manifold can be written as

$$p = \frac{1}{2d}X_1^{2d} + \frac{1}{2d}X_2^{2d} + \frac{1}{d\sqrt{c}}X_1^d X_2^d + \tilde{W}(\frac{X_1 X_2}{c^{1/d}}, X_3, X_4, X_5), \tag{5.1}$$

where c is a one of the parameters which characterize the complex structures. After suitable reparametrization, one obtains the desirable form of [18] which can represent the embedding of the Seiberg-Witten curve in terms of the ALE space. If the $K3$ fibre admits the elliptic fibration, we have more $(-1,1)$ state written in the form $\left|j^{-\frac{2d}{n}}\right\rangle_{(\mathrm{a,c})}$ with the integer $n > 2$.

Let us remind that our orbifold example W_2 of (4.9) gives us a new set of relations of twist orerators, or equivalently a new set of Mori vectors. This fact suggests the interesting possibility that after the appropriate orbifoldization, we will get certain Calabi-Yau manifolds in which the method of geometric engineering proposed in [19] will adequately work.

As a simple and interesting example, consider the superpotential

$$W = X_1^{12} + X_2^{12} + X_3^6 + X_4^6 + X_5^2. \tag{5.2}$$

This model corresponds to the Calabi-Yau manifold whose hypersurface is embedded in $WCP_{(1,1,2,2,6)}[12]^2_{-252}$. This hypersurface admits $K3$-fibration. This model

contains two $(-1,1)$ states $|j^{-2}\rangle_{(\mathrm{a,c})}$ and $|j^{-6}\rangle_{(\mathrm{a,c})}$. Their toric data are $(0,0,0,0)$ and $(0,-1,-1,-3)$, respectively. The dual polyhedron Δ_W^* consists of the following vertices

$$\begin{aligned} \nu_1^* = (-1,-2,-2,-6) \quad &, \quad \nu_2^* = (1,0,0,0), \quad \nu_3^* = (0,1,0,0), \\ \nu_4^* = (0,0,1,0) \qquad\quad &, \quad \nu_5^* = (0,0,0,1), \quad \nu_6^* = (0,-1,-1,-3), \end{aligned}$$

and we have two Mori vectors $l_1^{(1)} = (0;1,1,0,0,0,-2,0)$ and $l_1^{(2)} = (-6;0,0,1,1,3,1,0)$ (the last entry 0 is unnecessary at present, but we have added it for later use). These Mori vectors $l_1^{(1)}$ and $l_1^{(2)}$ correspond to the two relations of the $U(1)$ twist operators $(j^{-6})^2 = j^{-12}$ and $j^{-6} = (j^{-1})^6$ respectively, as noted in the previous section. Due to the these Mori vectors, the method of geometric engineering in [19] does not work as it stands. However after the appropriate orbifoldization, we can apply that method to obtain a quantum field theory.

Consider the $\mathbb{Z}_2$ orbifoldization

$$X_2 \sim -X_2, \quad X_5 \sim -X_5. \tag{5.3}$$

We then obtain the orbifoldized superpotential

$$W' = Y_1^{12} + Y_2^6 + Y_3^6 + Y_4^6 + Y_2 Y_5^2. \tag{5.4}$$

The corresponding hypersurface is embedded in $WCP_{(1,2,2,2,5)}[12]^4_{-180}$. Note that the corresponding hypersurface does not admit $K3$-fibration. But the hypersurface which corresponds to the following transposed potential

$$\tilde{W}' = \tilde{Y_1}^{12} + \tilde{Y_2}^6 \tilde{Y_5} + \tilde{Y_3}^6 + \tilde{Y_4}^6 + \tilde{Y_5}^2 \tag{5.5}$$

does admit $K3$-fibration, since that hypersurface is embedded in $WCP_{(1,1,2,2,6)}[12]$. The original Calabi-Yau manifold is embedded in the same weighted projective space. This is the general property for our orbifoldization that the transposed hypersurface of the orbifoldized one is embedded in the same weighted projective space as the original hypersurface is embedded in. This fact is important. To obtain a quantum field theory, we have only to consider the local mirror symmetry of [19]. So at least one of the hypersurfaces corrresponding to W' and $\tilde{W}'$ should have the $K3$-fibration structure.

At the orbifold point, we still have the two $(-1,1)$ states $|j^{-2}\rangle_{(\mathrm{a,c})}$ and $|j^{-6}\rangle_{(\mathrm{a,c})}$ so that the dual polyhedron Δ_W^* is unchanged. However once we consider the orbifoldized potential (5.4), we obtain a new set of operators $\{\tilde{\rho}_i\}$ satisfying

$$\tilde{\rho}_1{}^{12} = \tilde{\rho}_2{}^{6}\tilde{\rho}_5 = \tilde{\rho}_3{}^{6} = \tilde{\rho}_4{}^{6} = \tilde{\rho}_5{}^{2} = \text{identity}. \tag{5.6}$$

Since the new relation of operators $(\tilde{\rho}_2^6\tilde{\rho}_5)^2 = \tilde{\rho}_2^{12}\tilde{\rho}_5^2 = $ identity holds, we have a new toric data $\nu_7^* = (2,0,0,-1)$ which corresponds to $\tilde{\rho}_2^{12} = $ identity. The insertion of ν_7^* into the dual polyhedron Δ_W^* implies the toric blow-up. As a result, we obtain a new set of Mori vectors

$$l_2^{(1)} = (0;1,1,0,0,0,-2,0) = l_1^{(1)}, \tag{5.7}$$

$$l_2^{(2)} = (0;0,-2,0,0,1,0,1), \tag{5.8}$$

$$l_2^{(3)} = (-6;0,2,1,1,2,1,-1). \tag{5.9}$$

The orbifold relation is $l_1^{(2)} = l_2^{(2)} + l_2^{(3)}$. Using $l_2^{(1)}$ and $l_2^{(2)}$, we can obtain a quantum field theory by the geometric engineering of [19].

6 Discussions

In this paper, we have given the toric interpretation of Landau-Ginzburg models. This interpretation gives us the insight and the useful techniques to resolve the twisted sector problem. The orbifold construction is a powerful tool for solving the twisted sector problem. Using mirror symmetry, we can study somewhat large class of the Kähler moduli space. Our construction shows that through the orbifold point the different Landau-Ginzburg string vacua are connected. It is discussed that a relation of twist operators for orbifoldization have a natural interpretation as a Mori vector.

It is natural to expect that Landau-Ginzburg models with $c = 12$ correspond to Calabi-Yau fourfolds. Recently, Calabi-Yau fourfolds are studied in the context of F-theory [21, 22], i.e. a F-theory on a Calabi-Yau fourfold is a dual candidate for a heterotic string on a Calabi-Yau threefold. Our results of correspondence between the Landau-Ginzburg model and toric geometry can be easily extended to Calabi-Yau fourfolds and give us some insight for those theories.

In [23] it is pointed out that for Calabi-Yau fourfolds on which F-theory can compactify, the same difficulties as twisted sector problem for threefolds may occur when one considers the non-perturbative superpotential [26]. We hope that our approach for solving the twisted sector problem will be useful to improve the unexpected situations of Calabi-Yau fourfolds.

Recently, some attention has been paid to the extremal transition [24]. It is interesting to consider the relation between our orbifold construction and such extremal transition. In the context of F-theory, the 4-cycle transition is studied. For example [25], the Fermat hypersurface in $WCP_{(1,2,2,3,4)}[12]^2_{74}$ has the 4-cycle transition by which this theory transforms to the Fermat hypersurface in $WCP_{(1,1,1,1,2)}[6]^1_{103}$. It is easy to see that this transition can not be considered as our orbifold construction. So, it is interesting to consider whether our orbifold construction gives us a new class of examples as extremal transition.

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